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On the Trace Map for Products of Matrices Associated with Substitutive Sequences

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In a recent article, M. Kolář and M. K. Ali study the polynomial trace map for products of matrices associated with substitutive sequences on a two-letter alphabet, the existence of which has been proved by J.-P. Allouche and J. Peyrière. Computer calculations led them to conjecture some divisibility properties of the involved polynomials. The present work explains mathematically why it is so.

KEY WORDS: Trace map; substitutive sequences; automatic sequences; free groups.

The discovery of quasicrystals[1] gave rise to many theoretical studies of ordered, but noncrystallographic, systems of atoms (see, for example, refs. 2 and 3). One way of generating such one-dimensional systems is to use substitutions operating on a finite alphabet. Among these substitutions, those which act upon a two-letter alphabet are particuliarly important and convenient, due to the existence of a general theorem yielding a recursion formula for the traces of certain products of transfer matrices.[4] In a recent article, Kolář and Ali[5] were led by symbolic calculations on the computer to conjecture a certain divisibility property of the polynomials which govern such a recursion. In the present work, this conjecture is proved to hold. In addition, the use of the proper mathematical tools simplifies and considerably shortens certain results of ref. 5 while generalizing them to a larger framework. Indeed, a substitution on a two-letter alphabet (a, b), viewed as a homomorphism of the monoid of words over this alphabet into itself, is a particular endomorphism of the free group F generated by a and

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b. Besides, we think that the use of endomorphisms of free groups instead of mere substitutions could lead to new useful models.

Let us introduce some notations.

1. If G_1 and G_2 are two groups, $\mathrm{Hom}(G_1, G_2)$ denotes the set of homomorphisms from G_1 to G_2.

2. If σ and τ are elements of $\mathrm{Hom}(F, F)$, we set $\sigma\tau = \tau \circ \sigma$ (where $\circ$ denotes the composition of functions).

3. If K is a commutative field, $SL_2(K)$ denotes the set of 2×2 matrices with determinant 1 and the entries of which are in K.

4. An element φ of $\mathrm{Hom}(F, SL_2(K))$ is uniquely determined by the couple $(\varphi(a), \varphi(b))$ of elements of $SL_2(K)$.

5. Let us denote by T the following map from $\mathrm{Hom}(F, SL_2(K))$ to K^3: $T(\varphi) = (\mathrm{tr}\ \varphi(a), \mathrm{tr}\ \varphi(b), \mathrm{tr}\ \varphi(ab))$, where tr stands for the trace.

6. $\mathbb{Z}[x, y, z]$ denotes the set of polynomials in the variables x, y, and z, the coefficients of which are integers.

In these conditions we have the following results.

Theorem 1. For any $\sigma \in \mathrm{Hom}(F, F)$, there exists a unique $\Phi_\sigma \in (\mathbb{Z}[x, y, z])^3$ such that, for any K, and for any $\varphi \in \mathrm{Hom}(F, SL_2(K))$, we have

$$T(\varphi \circ \sigma) = \Phi_\sigma(T(\varphi))$$

Proof. The existence of Φ_σ is a mere reformulation of the theorem in ref. 4, which results from repetitive applications of the Cayley–Hamilton theorem. The uniqueness results from the fact that the triple $(\mathrm{tr}\ A, \mathrm{tr}\ B, \mathrm{tr}\ AB)$ can assume any value in $\mathbb{C}^3$ for suitable A and B in $SL_2(\mathbb{C})$.

Corollary 1. For σ and τ in $\mathrm{Hom}(F, F)$, we have $\Phi_{\sigma\tau} = \Phi_\sigma \circ \Phi_\tau$.

Proof. If $\varphi \in \mathrm{Hom}(F, SL_2(K))$, we have

$$T(\varphi \circ (\sigma\tau)) = T((\varphi \circ \tau) \circ \sigma) = \Phi_\sigma(T(\varphi \circ \tau)) = \Phi_\sigma \circ \Phi_\tau(T(\varphi))$$

and the corollary results from the uniqueness of $\Phi_{\sigma\tau}$.

Corollary 2. If σ is an automorphism of F, then the jacobian of Φ_σ is either 1 or -1.

Proof. This results from the chain rule and from the fact that $\det \Phi'_\sigma$ is a polynomial with integral coefficients.

In particular, this corollary explains why the maps associated with all substitutions (14) in ref. 5 are volume preserving.

Theorem 2. Let $\lambda(x, y, z) = x^2 + y^2 + z^2 - xyz - 4$. Then, for any $\sigma \in \mathrm{Hom}(F, F)$, there exists a polynomial $Q_\sigma \in \mathbb{Z}[x, y, z]$ such that $\lambda \circ \Phi_\sigma = \lambda \cdot Q_\sigma$.

Proof. For A and B in $SL_2(\mathbb{C})$, one has $\lambda(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = 0$ if and only if A and B have a common eigenvector. This can be seen as a corollary of Fricke's formula. One can also prove this assertion by observing that, in a suitable base, A and B assume the forms

$$\begin{pmatrix} \operatorname{tr} A & +1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \operatorname{tr} B & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$$

respectively, if A and B have no common eigenvector.

If $\varphi \in \mathrm{Hom}(F, SL_2(\mathbb{C}))$ is such that $\lambda(T(\varphi)) = 0$, then $\varphi(a)$ and $\varphi(b)$ have a common eigenvector, and so have $\varphi \circ \sigma(a)$ and $\varphi \circ \sigma(b)$. Therefore, $\lambda(T(\varphi \circ \sigma)) = 0$. In other terms, $\lambda(T(\varphi)) = 0$ implies $\lambda(\Phi_\sigma(T(\varphi))) = 0$. But, as we observed it previously, $T\varphi$ can be an arbitrary point in $\mathbb{C}^3$. Therefore $\lambda \circ \Phi_\sigma$ is divisible by λ in $\mathbb{Z}[x, y, z]$.

Proposition 1. For σ and τ in $\mathrm{Hom}(F, F)$, we have $Q_{\sigma\tau} = Q_\sigma \circ \Phi_\tau \cdot Q_\tau$.

Proof.

$$\lambda \circ \Phi_{\sigma\tau} = (\lambda \circ \Phi_\sigma) \circ \Phi_\tau = (\lambda \cdot Q_\sigma) \circ \Phi_\tau = \lambda \cdot Q_\tau \cdot Q_\sigma \circ \Phi_\tau$$

Corollary 1 and Proposition 1 extend Theorems 2 and 3 of ref. 5.

Proposition 2. For $\sigma \in \mathrm{Hom}(F, F)$, we have $Q_\sigma(0, 0, 0) = 0$ or 1.

Proof. Take

$$\varphi(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(b) = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Corollary 3. If σ is an automorphism of F, then $Q_\sigma = 1$.

Proof. We have

$$1 = Q_{\sigma^{-1}\sigma} = Q_{\sigma^{-1}} \circ \Phi_\sigma \cdot Q_\sigma$$

So Q_σ and $Q_{\sigma^{-1}} \circ \Phi_\sigma$ are nonzero constants, and therefore are identically 1.

This accounts for all examples (14) in ref. 5.

Example. If σ is the Fibonacci substitution $[\sigma(a) = ab,\ \sigma(b) = a]$, then σ is invertible $[\sigma^{-1}(a) = b,\ \sigma^{-1}(b) = b^{-1}a]$. Therefore, $Q_\sigma = 1$, which is exactly the relation discovered by Kohmoto *et al.*[6] and by Ostlund *et al.*[7]

Remark. This raises several questions:

1. Does $Q_\sigma = 1$ imply that σ is invertible?
2. How to describe the equivalence relation $Q_\sigma = Q_\tau$?
3. How to describe the set of polynomials Q_σ's?

Concerning question 2, if σ_1 and σ_2 are automorphisms of F and if Φ_{σ_2} is the identity, then $Q_{\sigma_1 \tau \sigma_2} = Q_\tau$ for any τ.

Concerning question 3, the set of polynomials Q_σ is invariant under permutations of variables: if $\sigma(a) = b$ and $\sigma(b) = a$, then $\Phi_\sigma(x, y, z) = (y, x, z)$, and, if $\sigma(a) = a^{-1}$ and $\sigma(b) = ab$, then $\Phi_\sigma(x, y, z) = (x, z, y)$.

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